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Intuitionistic Fuzzy Complex Subgroups with Respect to Norms (T,S)

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Abstract

In our work in this paper, we define intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S and investigate some properties of them in detail. Next, we obtain some results about them and give some relationships between them. Later, we introduce the inverse, composition, intersection and normality of them and we prove some basic new results and present some properties of them such that the inverse and composition of two intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S will be intuitionistic complex fuzzy subgroups with respect to t-norm T and s-norm S. Also we consider and give some characterizations of them. Finally, we discuss them under group homomorphisms and investigate some related properties such that the image and preimage of two intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S will be intuitionistic complex fuzzy subgroups with respect to t-norm T and s-norm S.

Keywords: Group theory, Theory of fuzzy sets, Intuitionistic fuzzy complex groups, Norms, Homomorphisms, Intersection.

1 | Introduction

In mathematics, fuzzy sets (uncertain sets) are somewhat like sets whose elements have degrees of membership. The concept of fuzzy sets was introduced by Zadeh [1] in 1965. Atanassov [2] innovated the theory of Intuitionistic Fuzzy Sets (IFS) as a powerful extension of classical fuzzy sets. This particular theory has been a great source of inspiration for many mathematicians in various scientific fields like decision making problems [3] and medical diagnosis determination [4]. Roenfeld [5] started the investigation of fuzzy subgroups and found numerous essential properties of this concept. Biswas [6] started the conception of intuitionistic fuzzy subgroups in 1997. A new concept of complex fuzzy sets was presented by Ramot et al. [7]. The extension of fuzzy sets to complex fuzzy sets is comparable to the extension of real numbers to complex numbers. The more development of complex fuzzy sets can be viewed in [8]. Alkouri and Salleh [9] gave the idea of complex intuitionistic fuzzy subsets and enlarge the basic properties of this phenomena. This concept became more effective and useful in scientific field because it deals with degree of membership and non-membership in complex plane. They also initiated the concept of complex intuitionistic fuzzy relation and developed fundamental

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operation of complex IFSs in [10]. Al-Husban and Salleh [11] introduced the concept of complex fuzzy subgroups in 2016. Ali and Tamir [12] innovated the notion complex intuitionistic fuzzy classes in 2016. The author by using norms, investigated some properties of fuzzy algebraic structures [13–15]. In Section 2, we recall the elementary notions which will be needed in the sequel. Next, in Section 3, we define intuitionistic fuzzy complex subgroups with respect to t-norm T and t-conorm S (in short, *IFCN(G)*) of *G* and investigate properties of them as *Propositions 2* and *3*. Later, in Section 4, we introduce composition, inverse and intersection of two elements $A, B\in IFCN(G)$ and we prove that $AoB\in IFCN(G)$ and $A\cap B\in IFCN(G)$ under some conditions. Also in Section 5, we define normality of two elements $A,B\in IFCN(G)$ and discuss some properties of them. Finally, in Section 6, we investigate image and pre image of them under group homomorphisms.

2 | Preliminaries

We recall first the elementary notions which play a key role for our further analysis. This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [2, 7, 9], [16–21].

Definition 1. A group is a non-empty set G on which there is a binary operation (a, b) as ab such that

I. If a and b belong to G then ab is also in G (closure).
II. $a(bc)=(ab)c$ for all $a, b, c \in G$ (associativity).
III. There is an element $e_G\in G$ such that $ae_G=e_Ga=a$ for all $a\in G$ (identity).
IV. If $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1}=a^{-1}a=e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a,b \in G$.

Remark 1: There are two standard notations for the binary group operation: either the additive notation, that is $a, b)=a+b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $a, b)=ab$ for which the identity is denoted by e.

Definition 2. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$.

Definition 3. For sets X, Y and Z, $f=(f_1, f_2) :X\rightarrow Y\times Z$ is called a complex mapping if $f_1:X\rightarrow Y$ and $f_2 :X \rightarrow Z$ are mappings.

Definition 4. Let X be a nonempty set. A complex mapping $A=(\mu_A, \vartheta_A) :X\rightarrow[0,1]\times[0,1]$ is called an IFS in X if $\mu_A+ \vartheta_A \leq 1$ where the mappings $\mu_A :X\rightarrow[0,1]$ and $\vartheta_A :X\rightarrow[0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\vartheta_A(x)$) for each $x\in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x)=(0,1)$, respectively. We will denote the set of all *IFSs* in X as $IFS(X)$.

Definition 5. Let X be a nonempty set and let $A=(\mu_A, \vartheta_A)$ and $A=(\mu_B, \vartheta_B)$ be *IFSs* in X. Then

I. $A\subset B$ iff $\mu_A \leq \mu_B$ and $\vartheta_A \geq \vartheta_B$.
II. $A=B$ iff $A\subset B$ and $B\subset A$.

Definition 6. Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x))|x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x)$ lies within the unit circle in the complex plane. We shall assume that is $\mu_A(x)$ will be represented by $r_{A(x)}e^{iw_A(x)}$ where $i=\sqrt{-1}$, and $r : X \rightarrow [0,1]$ and $w : X \rightarrow [0,2\pi]$. Note that by setting

$w(x)=0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1=r_1e^{w_1}$, and $\mu_2=r_2e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \le \mu_2$, we mean $r_1 \le r_2$ and $w_1 \le w_2$.

Definition 7. A complex IFS A, defined on a universe of discourse U, is characterized by membership and non-membership functions $\mu_A(x)$ and $\gamma_A(x)$, respectively, that assign any element $x \in U$ a complex-valued grade of both membership and non-membership in S. By definition, the values of $\mu_A(x)$ and $\gamma_A(x)$ and their sum may receive all lying within the unit circle in the complex plane, and are on the form $\mu_A(x)=r_A(x)e^{iw_{\mu_A}(x)}$ for membership function in S and $\gamma_A(x)=k_A(x)e^{iw_{\gamma_A}(x)}$ for non-membership function in A, where $i=\sqrt{-1}$, each of $r_A(x)$ and $k_A(x)$ are real-valued and both belong to the interval $[0,1]$ such that $0\le r_A(x)+k_A(x)\le 1$ and $iw_{\mu_A}(x)$ and $iw_{\gamma_A}(x)$ are real-valued.

Definition 8. A t-norm T is a function $T:[0,1]\times[0,1]\rightarrow[0,1]$ having the following four properties:

I. $T(x,1)=x$ (neutral element).
II. $T(x,y)\le T(x,z)$ if $y\le z$ (monotonicity).
III. $T(x,y)=T(y,x)$ (commutativity).
IV. $T(x,T(y,z))=T(T(x,y),z)$ (associativity).

For all $x, y, z \in [0,1]$.

It is clear that if $x_1\ge x_2$ and $y_1\ge y_2$, then $T(x_1,y_1)\ge T(x_2,y_2)$.

Example 1.

I. Standard intersection T-norm $T_m(x,y)=min\{x,y\}$
II. Bounded sum T-norm $T_b(x,y)=max\{0,x+y-1\}$.
III. Algebraic product T-norm $T_p(x,y)=xy$.

IV. Drastic T-norm.

$$T_D(x,y)=\begin{cases} y, & \text{if } x=1,\\ x, & \text{if } y=1,\\ 0, & \text{otherwise.}\end{cases}$$

V. Nilpotent minimum T-norm.

$$T_{nM}(x,y)=\begin{cases} \min\{x,y\}, & \text{if } x+y>1,\\ 0, & \text{otherwise.}\end{cases}$$

VI. Hamacher product T-norm.

$$T_{H_0}(x,y)=\begin{cases} 0, & \text{if } x=y=0,\\ \dfrac{xy}{x+y-xy}, & \text{otherwise.}\end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y)\le T(x,y)\le T_m(x,y)$ for all $x, y\in[0,1]$.

Recall that t-norm T will be idempotent if for all $x\in[0,1]$, we have $T(x,x)=x$.

Lemma 1. Let T be a t-norm. Then

$$T(T(x,y),T(w,z))=T(T(x,w),T(y,z)), \textit{ for all } x,y,w,z\in[0,1].$$

Definition 9. An s-norm S is a function $S:[0,1]\times[0,1]\rightarrow[0,1]$ having the following four properties:

I. $S(x,0)=x$.
II. $S(x,y)\le S(x,z)$ if $y\le z$.

III. $S(x, y)=S(y, x)$.
IV. $S(x, S(y, z))=S(S(x, y), z)$.

For all $x, y, z \in [0,1]$.

We say that S is idempotent if for all $x \in [0,1]$, $S(x, x) = x$.

Example 2. The basic S-norms are

$$S_m(x, y) = \max\{x, y\},$$
$$S_b(x, y) = \min\{1, x + y\},$$

and

$$S_p(x, y) = x+y-xy,$$

For all $x, y \in [0,1]$.

S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

Lemma 2. Let S be a s-norm. Then $S(S(x, y), S(w, z)) = S(S(x, w), S(y, z))$, for all $x, y, w, z \in [0,1]$.

Proposition 1. Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

I. H is a subgroup of G.
II. $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 9. Let H be subgroup of group G. Then we say that H is normal subgroup of G if for all $g \in G$ and $h \in H$, we have that $ghg^{-1} \in H$.

Definition 10. Let G and H be any two groups and $f: G \to H$ be a function. Then f is called a homomorphism if $f(xy)=f(x)f(y)$ for all $x, y \in G$.

3 | Intuitionistic Fuzzy Complex Subgroups with Respect to Norms (t-Norm T and s-Norm S)

Definition 11. Let G be a group such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A = r_A e^{iw_A}$ be two complex fuzzy sets on G. An $A=(\mu_A, \vartheta_A) \in IFS(G)$ is said to be intuitionistic complex fuzzy subgroup with respect to norms(t-norm T and s-norm S) (in short, $IFCN(G)$) of G if

I. $R_A(xy) \geq T(r_A(x), r_A(y))$.
II. $r_A(x^{-1}) \geq r_A(x)$.
III. $W_A(xy) \geq \min\{w_A(x), w_A(y)\}$.
IV. $W_A(x^{-1}) \geq w_A(x)$.
V. $R_A(xy) \leq S(r_A(x), r_A(y))$.
VI. $R_A(x^{-1}) \leq r_A(x)$.
VII. $W_A(xy) \leq \max\{w_A(x), w_A(y)\}$.
VIII. $W_A(x^{-1}) \leq w_A(x)$.

For all $x, y \in G$.

Example 3. Let $G = \{0, a, b, c\}$ be the Klein's group. Every element is its own inverse, and the product of any two distinct non-identity elements is the remaining non-identity element. Thus the Klein 4-group admits the elegant presentation $a^2 = b^2 = c^2 = abc = 0$. Define $r_A : G \to [0, 1]$ by

$$r_A(x)=\begin{cases}0.75, & \text{if } x = a,\\ 0.65, & \text{if } x = b,\\ 0.55, & \text{if } x = c,\\ 0.45, & \text{if } x = 0,\end{cases}$$

and $w_A : G \to [0, 2\pi]$ by

$$w_A(x)=\begin{cases}0.45\pi, & \text{if } x = a,\\ 0.45\pi, & \text{if } x = b,\\ 0.55\pi, & \text{if } x = c,\\ 0.65\pi, & \text{if } x = 0.\end{cases}$$

$r_A : G \to [0, 1]$ by

$$r_A(x)=\begin{cases}0.25, & \text{if } x = a,\\ 0.35, & \text{if } x = b,\\ 0.45, & \text{if } x = c,\\ 0.55, & \text{if } x = 0,\end{cases}$$

and $w_A : G \to [0, 2\pi]$ by

$$w_A=\begin{cases}0.55\pi, & \text{if } x = a,\\ 0.55\pi, & \text{if } x = b,\\ 0.45\pi, & \text{if } x = c,\\ 0.35\pi, & \text{if } x = 0.\end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ and $S(a, b) = S_p(a, b) = a+b-ab$ for all $a, b \in [0, 1]$, then $A = (\mu_A, \vartheta_A) \in IFCN(G)$.

Proposition 2. Let $A = (\mu_A, \vartheta_A) \in IFCN(G)$ and T and S be idempotent. Then for all $x \in G$, and $n \geq 1$,

I. $A(e) \supseteq A(x)$.
II. $A(x^n) \supseteq A(x)$.
III. $A(x) = A(x^{-1})$.

Proof: As $\mu_A = r_A e^{iw_A} \in ICFN(G)$ so

I.

$$r_A(e) = r_A(xx^{-1}) \geq T(r_A(x), r_A(x^{-1})) \geq T(r_A(x), r_A(x)) = r_A(x),$$

and

$$w_A(e) = w_A(xx^{-1}) \geq \min\{w_A(x), w_A(x^{-1})\} \geq \min\{w_A(x), w_A(x)\} = w_A(x),$$

and then

$$\mu_A(e) = r_A(e)e^{iw_A(e)} \geq r_A(x)e^{iw_A(x)} = \mu_A(x). \quad \text{(a)}$$

Also

$$w_A(e) = w_A(xx^{-1}) \leq \max\{w_A(x), w_A(x^{-1})\} \leq \max\{w_A(x), w_A(x)\} = w_A(x),$$

and so

$$\vartheta_A(e) = r_A(e)e^{iw_A(e)} \leq r_A(x)e^{iw_A(x)} = \vartheta_A(x). \quad \text{(b)}$$

Now from *Eqs. (a)* and *(b)* we obtain that

$$A(e) = (\mu_A(e), \vartheta_A(e)) \supseteq (\mu_A(x), \vartheta_A(x)) = A(x).$$

II.

$$r_A(x^n) = r_A\left(\underbrace{x.x...x}_{n}\right) \geq T(\underbrace{r_A\ x), r_A\ x), ... r_A(x}_{n})) = r_A(x),$$

and

$$w_A(x^n) = w_A\left(\underbrace{x.x...x}_{n}\right) \geq \min\{\underbrace{w_A\ x), w_A\ x), ... w_A(x}_{n})\} = w_A(x),$$

and so

$$\mu_A\ x^n) = r_A\ x^n)e^{iw_A\ x^n)} \geq r_A\ x)e^{iw_A\ x)} = \mu_A\ x). \tag{a}$$

Also

$$r_A(x^n) = r_A\left(\underbrace{x.x...x}_{n}\right) \leq S\{\underbrace{r_A\ x), r_A\ x), ..., r_A(x}_{n})\} = r_A(x),$$

and

$$w_A(x^n) = w_A\left(\underbrace{x.x...x}_{n}\right) \leq \max\{\underbrace{w_A\ x), w_A\ x), ..., w_A(x}_{n})\} = w_A(x),$$

and

$$\vartheta_A\ x^n) = r_A\ x^n)e^{iw_A\ x^n)} \leq r_A\ x)e^{iw_A\ x)} = \vartheta_A\ x). \tag{b}$$

Now using *Eqs. (a)* and *(b)* give us

$$A\ x^n) = \left(\mu_A\ x^n), \vartheta_A\ x^n)\right) \supseteq \left(\mu_A\ x), \vartheta_A\ x)\right) = A\ x).$$

III. As

$$r_A\ x) = r_A\left(x^{-1}\right)^{-1} \geq r_A\left(x^{-1}\right) \geq r_A\ x),$$

and so $r_A\ x) = r_A\left(x^{-1}\right)$ and as

$$w_A\ x) = w_A\left(x^{-1}\right)^{-1} \geq w_A\left(x^{-1}\right) \geq w_A\ x).$$

Then $w_A\ x) = w_A\left(x^{-1}\right)$. Then

$$\mu_A\left(x^{-1}\right) = r_A\left(x^{-1}\right)e^{iw_A\left(x^{-1}\right)} = r_A\ x)e^{iw_A\ x)} = \mu_A\ x). \tag{a}$$

Now

$$r_A\ x) = r_A\left(x^{-1}\right)^{-1} \leq r_A\left(x^{-1}\right) \leq r_A\ x),$$

so $r_A\ x) = r_A\left(x^{-1}\right)$ and as

$$w_A\ x) = w_A\left(x^{-1}\right)^{-1} \leq w_A\left(x^{-1}\right) \leq w_A\ x),$$

so $w_A\ x) = w_A\left(x^{-1}\right)$. Then

$$\vartheta_A\left(x^{-1}\right) = r_A\left(x^{-1}\right)e^{iw_A\left(x^{-1}\right)} = r_A\ x)e^{iw_A\ x)} = \vartheta_A\ x). \qquad \text{(b)}$$

Thus from *Eqs. (a)* and *(b)* we give that

$$A(x^{-1}) = (\mu_A(x^{-1}), \vartheta_A(x^{-1})) = (\mu_A(x), \vartheta_A(x)) = A(x).$$

Proposition 3. Let $A = \ \mu_A, \vartheta_A) \in IFCN\ G)$ and T and S be idempotent. Then $A(xy)=A(y)$ if and only if $A(x)=A(e)$ for all $x, y \in G$.

Proof: As $A\ xy)=A\ y)$ for all $x, y \in G$ so if we let $y=e$, then we get that $A\ x)= A\ e)$.

Conversely, suppose that $A\ x)=A\ e)$ so from *Proposition 2*, we get that $A\ x) \supseteq A\ y)$ and $A\ x) \supseteq A\ xy)$. Then $r_A\ x) \geq r_A\ y)$ and $r_A\ x) \geq r_A\ xy)$ and $w_A\ x) \geq w_A\ y), w_A\ xy)$. Also $r_A\ x) \leq r_A\ y)$ and $r_A\ x) \leq r_A\ xy)$ and $w_A\ x) \leq w_A\ y)$ and $w_A\ x) \leq w_A\ xy)$.

Now

$$\begin{aligned} r_A\ xy) &\geq T\left(r_A\ x), r_A\ y)\right) \\ &\geq T\left(r_A\ y), r_A\ y)\right) \\ &= r_A\ y) = r_A\left(x^{-1}xy\right) \\ &\geq T\left(r_A\ x), r_A\ xy)\right) \\ &\geq T\left(r_A\ xy), r_A\ xy)\right) \\ &= r_A\ xy), \end{aligned}$$

and then

$$r_A\ xy) = r_A\ y). \qquad \text{(a)}$$

Also

$$\begin{aligned} w_A\ xy) &\geq \min\left\{w_A\ x), w_A\ y)\right\} \\ &\geq \min\left\{w_A\ y), w_A\ y)\right\} = w_A\ y) = w_A\left(x^{-1}xy\right) \\ &\geq \min\left\{w_A\ x), w_A\ xy)\right\} \\ &\geq \min\left\{w_A\ xy), w_A\ xy)\right\} \\ &= w_A\ xy), \end{aligned}$$

and then

$$w_A\ xy) = w_A\ y). \qquad \text{(b)}$$

Therefore from *Eqs. (a)* and *(b)* we obtain that

$$\mu_A\ xy) = r_A\ xy)e^{iw_A\ xy)} = r_A\ y)e^{iw_A\ y)} = \mu_A\ y). \qquad \text{(c)}$$

Also

$$\begin{aligned} r_A\ xy) &\leq S\left(r_A\ x), r_A\ y)\right) \\ &\leq S\left(r_A\ y), r_A\ y)\right) \\ &= r_A\ y) = r_A\left(x^{-1}xy\right) \\ &\leq S\left(r_A\ x), r_A\ xy)\right) \\ &\leq S\left(r_A\ xy), r_A\ xy)\right) \\ &= r_A\ xy), \end{aligned}$$

Then

$$r_A\ xy) = r_A\ y). \qquad \text{(d)}$$

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Also

$$\begin{aligned} w_A\ xy) &\leq \max\{w_A\ x), w_A\ y)\} \\ &\leq \max\{w_A\ y), w_A\ y)\} \\ &= w_A\ y) = w_A\left(x^{-1}xy\right) \\ &\leq \max\{w_A\ x), w_A\ xy)\} \\ &\leq \max\{w_A\ xy), w_A\ xy)\} \\ &= w_A\ xy), \end{aligned}$$

and then

$$w_A\ xy) = w_A\ y). \tag{e}$$

Therefore from *Eqs. (d)* and *(e)* we obtain that

$$\vartheta_A\ xy) = r_A\ xy)e^{iw_A\ xy)} = r_A\ y)e^{iw_A\ y)} = \vartheta_A\ y). \tag{f}$$

Now as *Eqs. (c)* and *(f)* we get that

$$A(xy) = (\mu_A(xy), \vartheta_A(xy)) = (\mu_A(y), \vartheta_A(y)) = A(y).$$

4 | Composition, Inverse and Intersection of IFCN(G)

Definition 12. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ such that $\mu_A=r_Ae^{iw_A}\in IFCN\ G)$ and $\vartheta_A= r_Ae^{iw_A}$ and $\mu_B=r_Be^{iw_B}$ and $\vartheta_B=r_Be^{iw_B}$. We define the composion of A and B as $A\ oB$ and for all $x\in G$ we have

$$A\ o\ B)\ x)=\left(\ \mu_A, \vartheta_A)\ o\ \ \mu_B, \vartheta_B)\right)\ x)=\left(\mu_{AoB}\ x), \vartheta_{AoB}\ x)\right)=$$
$$((r_A or_B)(x)e^{i(w_A\ o\ w_B)(x)}, (r_A or_B)(x)e^{i(w_A\ o\ w_B)(x)}.$$

Such that $r_A\ o\ r_B : G \to [0, 1]$ and $w_A o w_B : G \to [0, 2\pi]$ and $r_A\ o\ r_B : G \to [0, 1]$ and $w_A o\ w_B : G \to [0, 2\pi]$. Now define

$$r_A\ o\ r_B)\ x)=\begin{cases} \sup\limits_{x=ab} T\left(r_A\ a), r_B\ b)\right), & \text{if} \quad x=ab, \\ 0, & \text{if} \quad x \neq ab, \end{cases}$$

and

$$(w_A\ o\ w_B)\ x)=\begin{cases} \min\limits_{x=ab}\{w_A\ a), w_B\ b)\}, & \text{if} \quad x=ab, \\ 0, & \text{if} \quad x\neq ab, \end{cases}$$

and

$$(r_A\ o\ r_B)\ x) = \begin{cases} \inf\limits_{x=ab} S\left(r_A\ a), r_B\ b)\right), & \text{if} \quad x = ab, \\ 0, & \text{if} \quad x \neq ab, \end{cases}$$

and

$$(w_A\ o\ w_B)\ x)=\begin{cases} \max\limits_{x=ab}\{w_A\ a), w_A\ b)\}, & \text{if} \quad x=ab, \\ 0, & \text{if} \quad x\neq ab, \end{cases}$$

For all $x\in G$.

Proposition 4. Let $A^{-1}=(\mu_A^{-1}, \vartheta_A^{-1})\in IFS(G)$ be the inverse of $A= \mu_A, \vartheta_A)\in IFCN(G)$ such that for all $x\in G$:

$$A^{-1}(x)=(\mu_A^{-1}(x), \vartheta_A^{-1}(x))=(\mu_A(x^{-1}), \nu A(x^{-1}))=A(x^{-1}).$$

If T and S be idempotent then $A= \mu_A, \vartheta_A)\in IFCN\ G)$ if and only if A satisfies the following conditions:

I. $A\supseteq A\ o\ A$.
II. $A^{-1}= A$.

Proof: Let $x, y, z \in G$ with $x = yz$ and $A \in ICF\,N(G)$. Then

I.

$$r_A(x) = r_A(yz) \geq T(r_A(y), r_A(z)) = (r_A \text{ o } r_A)(x),$$

and

$$w_A(x) = w_A(yz) \geq \min\{w_A(y), w_A(z)\} = (w_A \text{ o } w_A)(x),$$

then

$$\mu_A(x) = r_A(x)e^{iw_A(x)} \geq (r_A \text{ o } r_A)(x)e^{i(w_A o w_A)(x)} = (\mu_{AoA})(x). \quad (a)$$

Also

$$r_A(x) = r_A(yz) \leq S(r_A(y), r_A(z)) = (r_A \text{ o } r_A)(x),$$

and

$$w_A(x) = w_A(yz) \geq \min\{w_A(y), w_A(z)\} = (w_A \text{ o } w_A)(x),$$

then

$$\vartheta_A(x) = r_A(x)e^{iw_A(x)} \geq (r_A \text{ o } r_A)(x)e^{i(w_A o w_A)(x)} = (\vartheta_{AoA})(x). \quad (b)$$

Thus from *Eqs. (a)* and *(b)* we get that

$$A(x) = (\mu_A(x), \vartheta_A(x)) \supseteq (\mu_{AoA})(x), \vartheta_{AoA})(x)) = (A \text{ o } A)(x),$$

and then $A \supseteq A\ oA$.

II. As *Proposition 2*, we have that $A^{-1}(x) = A(x^{-1}) = A(x)$ for all $x \in G$. Thus $A^{-1} = A$.

Conversely, let $A \supseteq A \text{ o } A$ and $A^{-1} = A$ and $x, y, z \in G$ with $x = yz$. Since $A \supseteq A \text{ o } A$ so $r_A(x) \geq (r_A \text{ o } r_A)(x)$ and then

$$r_A(yz) = r_A(x) \geq (r_A o r_A)(x) = \sup_{x=ab} T(r_A(y), r_B(z)) \geq T(r_A(y), r_A(z)). \quad (a)$$

$w_A(x) \geq (w_A \text{ o } w_A)(x)$ and thus,

$$w_A(yz) = w_A(x) \geq (w_A \text{ o } w_A)(x) = \min_{x=yz}\{w_A(y), w_B(z)\} \geq \{w_A(y), w_A(z)\}. \quad (b)$$

$r_A(x) \leq (r_A \text{ o } r_A)(x)$ and then,

$$r_A(yz) = r_A(x) \leq (r_A o r_A)(x) = \inf_{x=yz} S(r_A(y), r_B(z)) \leq S(r_A(y), r_A(z)). \quad (c)$$

$w_A(x) \leq (w_A \text{ o } w_A)(x)$ and so

$$w_A(yz) = w_A(x) \leq (w_A o w_A)(x) = \max_{x=yz}\{w_A(y), w_A(z)\} \leq \{w_A(y), w_A(z)\}. \quad (d)$$

As $A^{-1} = A$ so,

$$r_A(x^{-1}) = r_A^{-1}(x) = r_A(x). \quad (e)$$

$$r_A(x^{-1}) = r_A^{-1}(x) = r_A(x). \quad (f)$$

$$w_A(x^{-1}) = w_A^{-1}(x) = w_A(x). \quad (g)$$

$$w_A(x^{-1}) = w_A^{-1}(x) = w_A(x). \quad (h)$$

Thus from *Eqs. (a)-(h)*, we get that $A \in IFCN(G)$.

Corollary 1. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ and G be commutative group. Then $A\ o\ B\in IFCN(G)$ if and only if $A\ o\ B = B\ o\ A$.

Proof: If $A, B, A\ o\ B\in IFCN(G)$, then from *Proposition 4* we get that $A^{-1} = A, B^{-1} = B$ and $(BoA)^{-1} = B\ o\ A$. Now $A\ o\ B= A^{-1}\ o\ B^{-1} =(BoA)^{-1} =B\ o\ A$. conversely, since $A\ o\ B = B\ o\ A$ we have

$$(AoB)^{-1} =(BoA)^{-1} =A^{-1}\ o\ B^{-1} =A\ o\ B.$$

Also

$$(A\ o\ B)\ o\ (A\ o\ B) =A\ o\ (B\ o\ A)\ o\ B =A\ o\ (A\ o\ B)\ o\ B =(A\ o\ A)\ o\ (B\ o\ B)\subseteq A\ o\ B.$$

Now *Proposition 4* gives us that $A\ o\ B\in IFCN(G)$.

Definition 13. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ such that, $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)= r_A e^{iw_A}$ and $\mu_B= r_B e^{iw_B}$ and $\vartheta_B\ x)=r_B e^{iw_B}$. define the intersection of A and B as $A\cap B$ such that for all $x\in G$:

$$\begin{aligned}
&(A\cap B)(x) = (\ \mu_A, \vartheta_A) \cap\ \ \mu_B, \vartheta_B))(x)\\
&=(\mu_{A\cap B}(x), \nu A\cap B(x))\\
&=((r_A\cap r_B)(x)e^{i(w_A\cap w_B)(x)}, ((r_A\cap r_B)(x)e^{i\ w_A\cap\ w_B)(x)}.
\end{aligned}$$

Such that $r_A\cap r_B\colon G\to[0,1]$ and $w_A\cap w_B\colon G\to[0,2\pi]$ and $r_A\cap r_B\colon G\to[0,1]$ and $w_A\cap w_A : G\to[0,2\pi]$ define:

$$\begin{aligned}
&r_A\cap r_B)\ x)=T\big(r_A\ x), r_A\ x)\big),\\
&w_A\cap w_B)\ x)=\min\{w_A\ \ x), w_A\ \ x)\},\\
&r_A\cap r_B)\ x)= S\big(r_A\ x), r_B\ x)\big),\\
&w_A\cap w_B)\ x)=\max\{w_A\ \ x), w_B\ \ x)\},\\
&\text{for all } x\in G.
\end{aligned}$$

Proposition 5. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$. Then $A\cap B\in IFCN\ G)$.

Proof: Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ such that $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)= r_A e^{iw_A}$ and $\mu_B= r_B e^{iw_B}$ and $\vartheta_B\ x)= r_B e^{iw_B}$.

I. Let $g_1, g_2 \in G$. then

$$\begin{aligned}
&(r_A\cap r_B)(g_1g_2) = T\ (r_A(g_1g_2), r_B(g_1g_2))\\
&\geq T\ (T\ (r_A(g_1), r_A(g_2)), T\ (r_B(g_1), r_B(g_2)))\\
&= T\ (T\ (r_A(g_1), r_B(g_1)), T\ (r_A(g_2), r_B(g_2)))\ \text{(Lemma 1)}\\
&= T\ ((r_A\cap r_B)(g_1), (r_A\cap r_B)(g_2)),
\end{aligned}$$

and thus

$$(r_A\cap r_B)(g_1g_2) \geq T\ ((r_A\cap r_B)(g_1), (r_A\cap r_B)(g_2)).$$

II. If $g\in G$, then

$$(r_A\cap r_B)(g^{-1})=T\ (r_A(g^{-1}), r_B(g^{-1}))\geq T\ (r_A(g), r_B(g))=(r_A\cap r_B)(g),$$

and so $r_A\cap r_B)\big(g^{-1}\big)\geq r_A\cap r_B)\ g)$.

III. If $g\in G$, then

$$\begin{aligned}
&w_A\cap w_B)\ g_1g_2)= \min\ \{w_A\ g_1g_2), w_B\ g_1g_2)\}\\
&\geq \min\{\min\{w_A\ g_1),w_A\ g_2)\}, \min\{w_B\ g_1),w_B\ g_2)\}\}\\
&=\min\ \{\min\ \{w_A\ g_1),w_B\ g_1)\}, \min\ \{w_B\ g_2),w_B\ g_2)\}\}\\
&=\min\{\ w_A\cap w_B)\ g_1),\ \ w_A\cap w_B)\ g_2)\},
\end{aligned}$$

and so $w_A \cap w_B) g_1g_2) \geq min\{ w_A \cap w_B) g_1), w_A \cap w_B) g_2)\}$.

IV. Let g ∈ G, so

$$(w_A \cap w_B)(g^{-1}) = \min\{w_A(g^{-1}), w_B(g^{-1})\} \geq \min\{w_A\ g), w_B\ g)\} = (w_A \cap w_B)(g),$$

and so $(w_A \cap w_B)(g^{-1}) \geq (w_A \cap w_B)\ g)$.

V. Let $g_1, g_2 \epsilon G$. then

$$r_A \cap r_B)\ g_1g_2) = S\left(r_A\ g_1g_2), r_B\ g_1g_2)\right)$$
$$\leq S\left(S\left(r_A\ g_1), r_A\ g_2)\right), S\left(r_B\ g_1), r_B\ g_2)\right)\right)$$
$$= S\left(S\left(r_A\ g_1), r_B\ g_1)\right), S\left(r_A\ g_2), r_B\ g_2)\right)\right) \text{ Lemma 1)}$$
$$= S\left(r_A \cap r_B)\ g_1), r_A \cap r_B)\ g_2)\right),$$

and thus

$$(r_A \cap r_B)(g_1g_2) \leq S((r_A \cap r_B)(g_1), (r_A \cap r_B)(g_2)).$$

VI. If $g \in G$, then

$$(r_A \cap r_B)(g^{-1}) = S\ (r_A(g^{-1}), r_B(g^{-1})) \leq S\ (r_A(g), r_B(g)) = (r_A \cap r_B)(g),$$

and so $r_A \cap r_B)(g^{-1}) \leq r_A \cap r_B)\ g)$.

VII. Let $g_1, g_2 \epsilon G$. Then

$$w_A \cap w_B)\ g_1g_2) = \max\ \{w_A\ g_1g_2), w_B\ g_1g_2)\}$$
$$\leq \max\{\max\{w_A\ g_1), w_A\ g_2)\}, \max\{w_B\ g_1), w_B\ g_2)\}\}$$
$$= \max\ \{\max\ \{w_A\ g_1), w_B\ g_1)\}, \max\ \{w_A\ g_2), w_B\ g_2)\}\}$$
$$= \max\{\ w_A \cap w_B)\ g_1),\ w_A \cap w_B)\ g_2)\},$$

and so $w_A \cap w_B)\ g_1g_2) \leq max\{ w_A \cap w_B)\ g_1), w_A \cap w_B)\ g_2)\}$.

VIII. Let g ∈ G, so

$$w_A \cap w_B)(g^{-1}) = \max\{w_A(g^{-1}), w_B(g^{-1})\} \leq \max\{w_A\ g), w_B\ g)\} = w_A \cap w_B)\ g),$$

and so $(\dot{w}_A \cap \dot{w}_B)(g^{-1}) \leq (\dot{w}_A \cap \dot{w}_B)\ g)$.

Then above steps give us that $A \cap B \in IFCN(G)$.

Corollary 2. Let $I_n = \{1, 2, ..., n\}$. If $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \subseteq IFCN(G)$.

Then $A = \bigcap_{i \in I_n} A_i \in IFCN(G)$.

5 | Normality of ICFN(G)

Definition 14. Let $A = \mu_A, \vartheta_A) \in IFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$. We say that $A = \mu_A, \vartheta_A)$ is normal if for all $x, y \in G$, we have that $A(xyx^{-1}) = A(y)$ which means that $r_A(xyx^{-1}) = r_A(y)$ and $w_A(xyx^{-1}) = w_A(y)$ and $r_A(xyx^{-1}) = r_A(y)$ and $w_A(xyx^{-1}) = w_A\ y)$. We denote by $NIFCN(G)$ the set of all normal intuitionistic fuzzy complex subgroups with respect to norms (t-norm T and s-norm S).

Proposition 6. Let $A = \mu_A, \vartheta_A) \in NIFCN\ G)$ and $B = \mu_B, \vartheta_B) \in NIFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$. Then $A \cap B \in NICFN\ G)$.

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Proof: As *Proposition 5* we have that $A \cap B \in IFCN\ G)$. Let $x, y, \in G$ then

I. $(r_A \cap r_B)(xyx^{-1}) = T(r_A(xyx^{-1}), r_B(xyx^{-1})) = T(r_A(y), r_B(y)) = (r_A \cap r_B)(y)$.
II. $(w_A \cap w_B)(xyx^{-1}) = \min\{w_A(xyx^{-1}), w_B(xyx^{-1})\} = \min\{w_A\ y), w_B\ y) = (w_A \cap w_B)(y)$.
III. $(r'_A(\cap r'_B)(xyx^{-1}) = S(r'_A((xyx^{-1}), r'_B(xyx^{-1})) = S(r'_A((y), r'_B(y)) = (r'_A(\cap r'_B)(y)$.
IV. $(w'_A \cap w'_B)(xyx^{-1}) = \max\{w'_A(xyx^{-1}), w'_B(xyx^{-1})\} = \max\{w'_A\ y), w'_B\ y)\} = (w'_A\ w'_B)(y)$.

Then from above steps, we get that

$$(A \cap B)(xyx^{-1}) = (\mu_{A \cap B}(xyx^{-1}), \vartheta_{A \cap B}(xyx^{-1})) = (\mu_{A \cap B}(y), \vartheta_{A \cap B}(y)) = (A \cap B)(y).$$

And so $A \cap B \in NIFCN(G)$.

Corollary 3. Let $I_n = \{1, 2, \ldots, n\}$. If $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \subseteq NIFCN(G)$. Then $A = \bigcap_{i \in I_n} A_i \in NIFCN(G)$.

Definition 15. Let $A = \mu_A, \vartheta_A) \in NIFCN\ G)$ and $B = \mu_B, \vartheta_B) \in IFCN\ G)$ such that $A \subseteq B$. Then A is called normal of B, written $A \sqsubseteq B$, if

I. $r_A(xyx^{-1}) \geq T(r_A\ y), r_B\ x))$.
II. $w_A(xyx^{-1}) \geq \min\{w_A(y), w_B(x)\}$.
III. $r_A(xyx^{-1}) \leq S\{r_A\ y), r_B\ x)\}$.
IV. $w_A(xyx^{-1}) \leq \max\{w_A(y), w_B(x)\}$.

For all $x, y \in G$.

Proposition 7. If T and S be idempotent and $A = \mu_A, \vartheta_A) \in IFCN\ G)$, then $A \subseteq A$.

Proof: Let $x, y \in G$ and $A = \mu_A, \vartheta_A) \in IFCN\ G)$. Then

$$r_A(xyx^{-1}) \geq T\left(r_A\ xy), r_A(x^{-1})\right)$$
$$\geq T(r_A\ xy), r_A\ x)) \geq T\left(T(r_A\ x), r_A\ y)), r_A\ x)\right)$$
$$= T\left(T(r_A\ x), r_A\ x)), r_A\ y)\right) = T(r_A\ x), r_A\ y)) = T(r_A(y), r_A(x)),$$

and so

$$r_A(xyx^{-1}) \geq T(r_A\ y), r_A\ x)). \tag{1}$$

also

$$w_A(xyx^{-1}) \geq \min\{w_A\ xy), w_A(x^{-1})\}$$
$$= \min\{w_A\ xy), w_A\ x)\}$$
$$\geq \min\{\min\{w_A\ x), w_A\ y)\}, w_A\ x)\}$$
$$= \min\{\min\{w_A\ x), w_A\ x)\}, w_A\ y)\}$$
$$= \min\{w_A\ x), w_A\ y)\}$$
$$= \min\{w_A\ y), w_A\ x)\},$$

Then

$$w_A(xyx^{-1}) \geq \min\{w_A\ y), w_A\ x)\}. \tag{2}$$

Now

$$
\begin{aligned}
&r_A\left(xyx^{-1}\right) \leq S\left(r_A\ xy), r_A\left(x^{-1}\right)\right)\\
&\leq S\left(r_A\ xy), r_A\ x)\right)\\
&\leq S\left(S\left(r_A\ x), r_A\ y)\right), r_A\ x)\right)\\
&= S\left(S\left(r_A\ x), r_A\ x)\right), r_A\ y)\right)\\
&= S\left(r_A\ x), r_A\ y)\right)\\
&= S(r_A(y), r_A(x)),
\end{aligned}
$$

thus

$$r_A\left(xyx^{-1}\right) \leq S\left(r_A\ y), r_A\ x)\right). \tag{3}$$

Finally,

$$
\begin{aligned}
&w_A\left(xyx^{-1}\right) \leq \max\{w_A\ xy), w_A\left(x^{-1}\right)\}\\
&\leq \max\{w_A\ xy), w_A\ x)\}\\
&\leq \max\{\min\left\{w_A\ x), w_A\ y)\right\}, w_A\ x)\}\\
&= \max\{\min\{w_A\ x), w_A\ x)\}, w_A\ y)\\
&= \max\{w_A\ x), w_A\ y)\}\\
&= \max\{w_A\ y), w_A\ x)\},
\end{aligned}
$$

then

$$w_A\left(xyx^{-1}\right) \leq \min\{w_A\ y), w_A\ x)\}. \tag{4}$$

Then *Eqs. (1)-(4)* give us that $A \sqsubseteq A$.

Proposition 8. Let $A= \mu_A, \vartheta_A) \in NIFCN\ G)$ and $B= \mu_B, \vartheta_B) \in IFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$. If T and S be idempotent, then $A \cap B \sqsubseteq B$.

Proof: As *Proposition 6* we have that $A \cap B \in NIFCN(G)$. Let $x, y \in G$ then

$$
\begin{aligned}
&(r_A \cap r_B)\left(xyx^{-1}\right) = T((r_A(xyx^{-1}), r_B(xyx^{-1}))\\
&= T((r_A(y), r_B(xyx^{-1}))\\
&\geq T(rA(y), T(r_B(xy), r_B(x^{-1}))\\
&\geq T(r_A(y), T(T\ (r_B(x), r_B(y)), r_B(x)))\\
&= T(r_A(y), T(r_B(y), T\ (r_B(x), r_B(x))))\\
&= T\left(r_A\ y), T\left(r_B\ y), r_B\ x)\right)\right)\\
&= T\left(T\left(r_A\ y), r_B\ y)\right), r_B\ x)\right)\\
&= T((r_A \cap r_B)(y), r_B(x)),
\end{aligned}
$$

and then

$$(r_A \cap r_B)\left(xyx^{-1}\right) \geq T((r_A \cap r_B)(y), r_B(x)). \tag{5}$$

Also

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$$
\begin{aligned}
&(w_A \cap w_B)\left(xyx^{-1}\right) = \min\{(w_A(xyx^{-1}), w_B(xyx^{-1})\} \\
&= \min\{(w_A(y), w_B(xyx^{-1})\} \\
&\geq \min\{w_A(y), \min\{w_B(xy), w_B(x^{-1})\}\} \\
&\geq \min\{w_A(y), \min\{\min\{w_B(x), w_B(y)\}, w_B(x)\}\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), \mathrm{Min}\{w_B\ x), w_B\ x)\}\right\}\right\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), w_B\ x)\right\}\right\} \\
&= \min\{\min\{w_A(y), w_B(y)\}, w_B(x)\}\} \\
&= \min\{(w_A \cap w_B)(y), w_B(x)\},
\end{aligned}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_B(x)\}. \tag{6}$$

Now

$$
\begin{aligned}
&(r_A \cap r_B)\left(xyx^{-1}\right) = S((r_A(xyx^{-1}), r_B(xyx^{-1})) \\
&= S((r_A(y), r_B(xyx^{-1})) \\
&\leq S(r_A(y), S(r_B(xy), r_B(x^{-1})) \\
&\leq S\ (r_A(y), S(S(r_B(x), r_B(y)), r_B(x))) \\
&= S(r_A(y), S(r_B(y), S(r_B(x), r_B(x)))) \\
&= S\left(r_A\ y), S\left(r_B\ y), r_B\ x)\right)\right) \\
&= S\left(S\left(r_A\ y), r_B\ y)\right), r_B\ x)\right) \\
&= S((r_A \cap r_B)(y), r_B(x)),
\end{aligned}
$$

and then

$$(r_A \cap r_B)\left(xyx^{-1}\right) \leq S((r_A \cap r_B)(y), r_B(x)). \tag{7}$$

As

$$
\begin{aligned}
&(w_A \cap w_B)\left(xyx^{-1}\right) = \min\{w_A(xyx^{-1}), w_B(xyx^{-1})\} \\
&= \min\{w_A(y), w_B(xyx^{-1})\} \\
&\geq \min\{w_A(y), \min\{w_B(xy), w_B(x^{-1})\}\} \\
&\geq \min\{w_A(y), \min\{\min\{w_B(x), w_B(y)\}, w_B(x)\}\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), \min\{w_B\ x), w_B\ x)\}\right\}\right\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), w_B\ x)\right\}\right\} \\
&= \min\{\min\{w_A(y), w_B(y)\}, w_B(x)\}\} \\
&= \min\{(w_A \cap w_B)(y), w_B(x)\},
\end{aligned}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_B(x)\}. \tag{8}$$

Then *Eqs. (5)-(8)* mean that $A \cap B \sqsubseteq B$.

Proposition 9. Let $A= \mu_A, \vartheta_A) \in IFCN\ G)$ and $B= \mu_B, \vartheta_B) \in IFCN\ G)$ and $C= \mu_C, \vartheta_C) \in IFCN\ G)$ such that $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)=r_A e^{iw_A}$ and $\mu_B=r_B e^{iw_B}$ and $\vartheta_B\ x)= r_B e^{iw_B}$ and $\mu_C=r_C e^{iw_C}$ and $\vartheta_C\ x)= r_C e^{iw_C}$ Let T and S be idempotent and $A \sqsubseteq C$ and $B \sqsubseteq C$. Then $A \cap B \sqsubseteq C$.

Proof: From *Proposition 6* we get that $A \cap B \in ICFN(G)$. Now for all $x, y \in G$ we get that

$$
\begin{aligned}
&(r_A \cap r_B)(xyx^{-1}) = T(r_A(xyx^{-1}), r_B(xyx^{-1})) \\
&\geq T(T(r_A(y), r_C(x)), T(r_B(y), r_C(x))) \\
&= T(T(rA(y), r_B(y)), T(r_C(x), r_C(x))) \\
&= T(T(r_A(y), r_B(y)), r_C(x)) \\
&= T((r_A \cap r_B)(y), r_C(x)),
\end{aligned}
$$

and then

$$r_A \cap r_B)\left(xyx^{-1}\right) \geq T\left(r_A \cap r_B) y), r_C x)\right). \tag{9}$$

Also

$$
\begin{aligned}
&w_A \cap w_B)\left(xyx^{-1}\right) = \min\left\{w_A\left(xyx^{-1}\right), w_B\left(xyx^{-1}\right)\right\} \\
&\geq \min\left\{\min\left\{w_A y), w_C x)\right\}, \min\left\{w_B y), w_C x)\right\}\right\} \\
&= \min\left\{\min\left\{w_A y), w_B y)\right\}, \min\{w_C x), w_C x)\}\right\} \\
&= \min\left\{\min\left\{w_A y), w_B y)\right\}, w_C x)\right\} \\
&= \min\{(w_A \cap w_B)(y), w_C(x)),
\end{aligned}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_C(x)). \tag{10}$$

As

$$
\begin{aligned}
&(r_A \cap r_B)(xyx^{-1}) = S(r_A(xyx^{-1}), r_B(xyx^{-1})) \\
&\leq S(S(r_A(y), r_C(x)), S(r_B(y), r_C(x))) \\
&= S(S(r_A(y), r_B(y)), S(r_C(x), r_C(x))) \\
&= S(S(r_A(y), r_B(y)), r_C(x)) \\
&= S((r_A \cap r_B)(y), r_C(x)),
\end{aligned}
$$

so

$$r_A \cap r_B)\left(xyx^{-1}\right) \leq S\left(r_A \cap r_B) y), r_C x)\right). \tag{11}$$

Since

$$
\begin{aligned}
&(w_A \cap w_B)(xyx^{-1}) = \max\{w_A\left(xyx^{-1}\right), w_B\left(xyx^{-1}\right)\} \\
&\leq \max\{\max\{w_A y), w_C x)\}, \max\{w_B y), w_C x)\}\} \\
&= \max\{\max\{w_A y), w_B y)\}, \max\{w_C x), w_C x)\}\} = \max\{\max\left\{w_A y), w_B y)\right\}, \\
&w_C x)\} \\
&= \max\{(w_A \cap w_B)(y), w_C(x)).
\end{aligned}
\tag{12}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \leq \max\{(w_A \cap w_B)(y), w_C(x)). \tag{13}$$

Then as *Eqs. (9)-(13)* we get that $A \cap B \sqsubseteq C$.

Corollary 4. Let $I_n = \{1, 2, \ldots, n\}$. If $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \subseteq IFCN(G)$ such that $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \sqsubseteq B = \mu_B, \vartheta_B)$. Then $A = \bigcap_{i \in I_n} A_i \sqsubseteq B = \mu_B, \vartheta_B)$.

6 | Group Homomorphisms and IFCN(G)

Definition 16. Let $A = \mu_A, \vartheta_A) \in IFCN\ G)$ and $B = \mu_B, \vartheta_B) \in IFCN\ H)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$.

Let $\varphi : G \to H$ be a group homomorphism. Define:

$$\varphi\ A) = \varphi(\mu_A), \varphi(\vartheta_A)) = \left(\varphi(r_A e^{iw_A}), \varphi(r_A e^{iw_A})\right) = \left(\varphi(r_A) e^{i\varphi(w_A)}, \varphi(r_A) e^{i\varphi(w_A)}\right).$$

For all $h \in H$ define:

$$\varphi(r_A) : H \to [0,1] \text{ as } \varphi(r_A)(h) = \sup\{r_A(g) \mid g \in G, \varphi(g) = h\},$$
$$\varphi(w_A) : H \to [0,2\pi] \text{ as } \varphi(w_A)(h) = \sup\{w_A(g) \mid g \in G, \varphi(g) = h\},$$
$$\varphi(r_A) : H \to [0,1] \text{ as } \varphi(r_A)(h) = \inf\{r_A(g) \mid g \in G, \varphi(g) = h\},$$

nd

$$\varphi(w_A) : H \to [0,2\pi] \text{ as } \varphi(w_A)(h) = \inf\{w_A(g) \mid g \in G, \varphi(g) = h\}.$$

Also define

$$\varphi^{-1}\ B) = \left(\varphi^{-1}(\mu_B), \varphi^{-1}(\vartheta_B)\right) = \left(\varphi^{-1}(r_B e^{iw_B}), \varphi^{-1}(r_B e^{iw_B})\right) =$$
$$\left(\varphi^{-1}(r_B) e^{i\varphi^{-1}(w_B)}, \varphi^{-1}(r_B) e^{i\varphi^{-1}(w_B)}\right),$$

such that for all $g \in G$:

$$\varphi^{-1}(r_B) : G \to [0,1] \text{ as } \varphi^{-1}(r_B)(g) = r_B(\varphi\ g)),$$
$$\varphi^{-1}(r_B) : G \to [0,1] \text{ as } \varphi^{-1}(r_B)(g) = r_B(\varphi\ g)),$$
$$\varphi^{-1}(w_B) : G \to [0,2\pi] \text{ as } \varphi^{-1}(w_B)(g) = w_B(\varphi\ g)),$$
$$\varphi^{-1}(w_B) : G \to [0,2\pi] \text{ as } \varphi^{-1}\ w_B)\ g) = w_B\left(\varphi\ g)\right).$$

Proposition 10. Let $A = \mu_A, \vartheta_A) \in IFCN\ G)$ and H be a group. Suppose that $\varphi : G \to H$ is a group homomorphism. Then $\varphi(A) \in IFCN(H)$.

Proof: Let $\varphi\ A) = \varphi(\mu_A), \varphi(\vartheta_A)) = \left(\varphi(r_A) e^{i\varphi(w_A)}, \varphi(r_A') e^{i\varphi(w_A')}\right)$ and $h_1, h_2 \in H$ and $g_1, g_2 \in G$ such that $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\varphi(r_A)(h_1h_2) = \sup\{r_A(g_1g_2) \mid g_1 = \varphi(h_1), g_2 = \varphi(h_2)\}$$
$$\geq \sup\{T(r_A(g_1), rA(g_2)) \mid g_1 = \varphi(h_1), g_2 = \varphi(h_2)\}$$
$$= T(\sup\{r_A(g_1) \mid g_1 = \varphi(h_1)\}, \sup\{r_A(g_2) \mid g_2 = \varphi(h_2)\})$$
$$= T(\varphi(r_A)(h_1), \varphi(r_A)(h_2)),$$

and so

$$\varphi\ r_A)\ h_1h_2) \geq T\left(\varphi\ r_A)\ h_1), \varphi\ r_A)\ h_2)\right). \tag{14}$$

Let $g \in G$ and $h \in H$ such that $\varphi(g) = h$. Then

$$\varphi(r_A)(h^{-1}) = \sup\{r_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1}) = h^{-1}\}$$
$$\geq \sup\{rA(g) \mid g^{-1} \in G, \varphi^{-1}(g) = h^{-1}\}$$
$$= \sup\{rA(g) \mid g \in G, \varphi(g) = h\}$$
$$= \varphi(rA)(h),$$

and then

$$\varphi\ r_A)\left(h^{-1}\right) \geq \varphi\ r_A)\ h). \tag{15}$$

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\varphi(w_A)(h_1h_2) = \sup\{w_A(g_1g_2) \mid g_1 = \varphi(h_1), g_2 = \varphi(h_2)\}$$
$$\geq \sup\{\min\{w_A\ g_1), w_A\ g_2)\} \mid g_1 = \varphi(h_1), g_2 = \varphi(h_2)\}$$
$$= \min\{\sup\{w_A(g_1) \mid g_1 = \varphi(h_1)\}, \sup\{w_A(g_2) \mid g_2 = \varphi(h_2)\}\}$$
$$= \min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\},$$

and so

$$\varphi\ w_A)\ h_1h_2) \geq \min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \tag{16}$$

Let $g \in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\varphi(w_A)(h^{-1})=\sup\{w_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\}$$
$$\geq \sup\{w_A(g) \mid g^{-1} \in G, \varphi^{-1}(g)=h^{-1}\}$$
$$= \sup\{w_A(g) \mid g \in G, \varphi(g) = h\}= \varphi(w_A)(h),$$

then

$$\varphi\ w_A)\ h_1h_2)\geq\min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \quad (17)$$

et $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\varphi(r_A)(h_1h_2)=\inf\{r_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\}$$
$$\leq \inf\{T(r_A(g_1), r_A(g_2)) \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\}$$
$$=S(\inf\{r_A(g_1) \mid g_1 =\varphi(h_1)\},\inf\{r_A(g_2) \mid g_2 =\varphi(h_2)\})$$
$$=S(\varphi(r_A)(h_1), \varphi(r_A)(h_2)),$$

then

$$\varphi\ w_A)\ h_1h_2)\geq\min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \quad (18)$$

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\varphi(r_A)(h_1h_2)=\inf\{r_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\}$$
$$\leq \inf\{T(r_A(g_1), r_A(g_2)) \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\}$$
$$=S(\inf\{r_A(g_1) \mid g_1 =\varphi(h_1)\},\inf\{r_A(g_2) \mid g_2 =\varphi(h_2)\})$$
$$=S(\varphi(r_A)(h_1), \varphi(r_A)(h_2)),$$

and so

$$\varphi\ w_A)\ h_1h_2)\geq\min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \quad (19)$$

Let $\in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\varphi(r_A)(h^{-1})=\inf\{r_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\}$$
$$\leq\inf\{r_A(g) \mid g^{-1} \in G,\varphi^{-1}(g)=h^{-1}\}$$
$$=\inf\{r_A(g) \mid g \in G, \varphi(g) = h\}$$
$$=\varphi(r_A)(h),$$

and then

$$\varphi\ r_A)\left(h^{-1}\right)\leq\varphi\ r_A)\ h). \quad (20)$$

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\varphi(w_A)(h_1h_2)=\inf\{w_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\}$$
$$\leq \inf\{\max\{w_A\ g_1),w_A\ g_2)\} \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\}$$
$$= \max\{\inf\{w_A(g_1) \mid g_1=\varphi(h_1)\},\inf\{w_A(g_2) \mid g_2 =\varphi(h_2)\}\}$$
$$= \max\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\},$$

and so

$$\varphi\ w_A)\ h_1h_2)\leq\max\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \quad (21)$$

Let $g \in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\varphi(w_A)(h^{-1})=\inf\{w_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\}$$
$$\leq \inf\{w_A(g) \mid g^{-1} \in G,\varphi^{-1}(g)=h^{-1}\}$$
$$= \inf\{w_A(g) \mid g \in G, \varphi(g) = h\}$$
$$= \varphi(w_A)(h),$$

then

$$\varphi\ w_A)\left(h^{-1}\right)\leq\varphi\ w_A)\ h). \quad (22)$$

Therefore from *Eqs. (14)-(22)* we get that $\varphi(A) \in ICFN(H)$.

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Proposition 11. Let H be a group and $B= \mu_B, \vartheta_B)\in IFCN\ H)$ and $\varphi:G\rightarrow H$ is a group homomorphism. Then $\varphi^{-1}\ B)\in IFCN\ G)$.

Proof: Let $B= \mu_B, \vartheta_B)\in IFCN\ H)$ such that $\mu_B=r_Be^{iw_B}$ and $\vartheta_B\ x)= r_Be^{iw_B}$ and $\varphi^{-1}\ B)=\left(\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)}\right)$, $\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)})\big)$. Let $g_1,g_2\in G$. Then

$$\begin{aligned}&\varphi^{-1}(r_B)(g_1g_2) = r_B(\varphi(g_1g_2))\\&= r_B(\varphi(g_1)\varphi(g_2))\\&\geq T(r_B(\varphi(g_1)), r_B(\varphi(g_2)))\\&=T(\varphi^{-1}(r_B)(g_1), \varphi^{-1}(r_B)(g_2)),\end{aligned}\tag{23}$$

and so $\varphi^{-1}\ r_B)\ g_1g_2)\geq T\left(\varphi^{-1}\ r_B)\ g_1), \varphi^{-1}\ r_B)\ g_2)\right)$.

$$\begin{aligned}&\varphi^{-1}(r_B)(g_1g_2)=r_B(\varphi(g_1g_2))\\&=r_B(\varphi(g_1)\varphi(g_2))\\&\leq T(r_B(\varphi(g_1)), r_B(\varphi(g_2)))\\&=T(\varphi^{-1}(r_B)(g_1), \varphi^{-1}(r_B)(g_2)),\end{aligned}\tag{24}$$

and so $\varphi^{-1}\ r_B)\ g_1g_2)\leq S\left(\varphi^{-1}\ r_B)\ g_1), \varphi^{-1}\ r_B)\ g_2)\right)$.

$$\begin{aligned}&\varphi^{-1}(w_B)(g_1g_2) = w_B(\varphi(g_1g_2))\\&= w_B(\varphi(g_1)\varphi(g_2))\\&\geq \min\{w_B\left(\varphi\ g_1)\right), w_B\left(\varphi\ g_2)\right)\}\\&= \min\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}\end{aligned}\tag{25}$$

and so $\varphi^{-1}\ w_B)\ g_1g_2)\geq \min\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}$.

$$\begin{aligned}&\varphi^{-1}(w'_B)(g_1g_2) = w'_B(\varphi(g_1g_2))\\&= w'_B(\varphi(g_1)\varphi(g_2))\\&\leq \max\{w'_B\left(\varphi(g_1)\right), w'_B\left(\varphi(g_2)\right)\}\\&= \max\{\varphi^{-1}(w'_B)(g_1), \varphi^{-1}(w'_B)(g_2)\},\end{aligned}\tag{26}$$

so $\varphi^{-1}\ w_B)\ g_1g_2)\leq \max\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}$.

$$\varphi^{-1}\ r_B)\left(g^{-1}\right)=r_B\left(\varphi\left(g^{-1}\right)\right)=r_B\left(\varphi^{-1}\ g)\right)\geq r_B\left(\varphi\ g)\right)=\varphi^{-1}\ r_B)\ g),\tag{27}$$

$$\varphi^{-1}\ r_B)\left(g^{-1}\right)=r_B\left(\varphi\left(g^{-1}\right)\right)=r_B\left(\varphi^{-1}\ g)\right)\leq r_B\left(\varphi\ g)\right)=\varphi^{-1}\ r_B)\ g),\tag{28}$$

$$\varphi^{-1}\ w_B)\left(g^{-1}\right)=w_B\left(\varphi\left(g^{-1}\right)\right)=w_B\left(\varphi^{-1}\ g)\right)\geq w_B\left(\varphi\ g)\right)=\varphi^{-1}\ w_B)\ g),\tag{29}$$

$$\varphi^{-1}\ w_B)\left(g^{-1}\right)=w_B\left(\varphi\left(g^{-1}\right)\right)=w_B\left(\varphi^{-1}\ g)\right)\leq w_B\left(\varphi\ g)\right)= \varphi^{-1}\ w_B)\ g).\tag{30}$$

Let $g\in G$.

Thus *Eqs. (23)-(30)* give us that $\varphi^{-1}(B)\in ICFN(G)$.

Proposition 12. Let $A= \mu_A, \vartheta_A)\in NIFCN\ G)$ and H be a group. Suppose that $\varphi:G\rightarrow H$ is a homomorphism. Then $\varphi(A)\in NIFCN(H)$.

Proof: Using *Proposition 10*, we give that $\varphi(A)\in IFCN(H)$. Let $x, y\in H$ such that $\varphi(u)=x$ and $\varphi(w)=y$ with $u, w\in G$. Then

$$\begin{aligned}
&\varphi(r_A(xyx^{-1}))=\sup\{r_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\sup\{r_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\sup\{r_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\sup\{r_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\sup\{r_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(r_A(y)),
\end{aligned} \tag{31}$$

so $\varphi\left(r_A\left(xyx^{-1}\right)\right)=\varphi\left(r_A\ y\right)\big)$.

$$\begin{aligned}
&\varphi(w_A(xyx^{-1}))=\sup\{w_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\sup\{w_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\sup\{w_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\sup\{w_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\sup\{w_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(w_A(y)),
\end{aligned} \tag{32}$$

then $\varphi\left(w_A\left(xyx^{-1}\right)\right)=\varphi\left(w_A\ y\right)\big)$.

$$\begin{aligned}
&\varphi(r_A(xyx^{-1}))=\inf\{r_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\inf\{r_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\inf\{r_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\inf\{r_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\inf\{r_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(r_A(y)),
\end{aligned} \tag{33}$$

then $\varphi\left(r'_A(xyx^{-1})\right)=\varphi\left(r'_A(y)\right)$.

$$\begin{aligned}
&\varphi(w_A(xyx^{-1}))=\inf\{w_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\inf\{w_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\inf\{w_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\inf\{w_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\inf\{w_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(w_A(y)),
\end{aligned} \tag{34}$$

then $\varphi\left(w_A\ xyx^{-1}\right)\big)=\varphi\left(w_A\ y\right)\big)$.

Thus for all $x, y\in H$ and from *Eqs. (31)-(34)* we get that

$$\begin{aligned}
&\varphi(A)(xyx^{-1})=(\varphi(\mu_A)(xyx^{-1}), \varphi(\vartheta_A)(xyx^{-1}))\\
&=(\varphi(r_A)(xyx^{-1}))e^{i\varphi(w_A)(xyx^{-1})}, \varphi(r_A)(xyx^{-1})e^{i\varphi(w_A)(xyx^{-1})})\\
&=(\varphi(r_A)(y))e^{i\varphi(w_A)(y)}, \varphi(r_A)(y)e^{i\varphi(w_A)(y)})\\
&=(\varphi(\mu_A)(y), \varphi(\vartheta_A)(y))\\
&=\varphi(A)(y),
\end{aligned}$$

Then $\varphi(A)\in NICFN(H)$.

Proposition 13. Let H be a commutative group and $B=\ \mu_B, \vartheta_B)\in NIFCN(H)$. If $\varphi :G \to H$ be a group homomorphism, then $\varphi^{-1}(B)\in NIFCN(G)$.

Proof: From *Proposition 11*, we get that φ^{-1} B)∈IFCN G). Let x, y∈G then

$$\varphi^{-1}(r_B)(xyx^{-1})=r_B(\varphi(xyx^{-1}))$$
$$=r_B(\varphi(x)\varphi(y)\varphi(x^{-1}))$$
$$=r_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \tag{35}$$
$$=r_B(\varphi(y))$$
$$=\varphi^{-1}(r_B)(y),$$

and thus $\varphi^{-1}\ r_B)\left(xyx^{-1}\right)=\varphi^{-1}\ r_B)\ y)$.

$$\varphi^{-1}(w_B)(xyx^{-1}) = w_B(\varphi(xyx^{-1}))$$
$$=w_B(\varphi(x)\varphi(y)\varphi(x^{-1}))$$
$$=w_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \tag{36}$$
$$=w_B(\varphi(y))$$
$$=\varphi^{-1}(w_B)(y),$$

so $\varphi^{-1}\ w_B)\left(xyx^{-1}\right)=\varphi^{-1}\ w_B)\ y)$.

$$\varphi^{-1}(r_B)(xyx^{-1}) = r_B(\varphi(xyx^{-1}))$$
$$=r_B(\varphi(x)\varphi(y)\varphi(x^{-1}))$$
$$= r_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \tag{37}$$
$$= r_B(\varphi(y))$$
$$= \varphi^{-1}(r_B)(y),$$

then $\varphi^{-1}\ r_B)\left(xyx^{-1}\right)=\varphi^{-1}\ r_B)\ y)$.

$$\varphi^{-1}(w_B)(xyx^{-1}) = w_B(\varphi(xyx^{-1}))$$
$$=w_B(\varphi(x)\varphi(y)\varphi(x^{-1}))$$
$$= w_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \tag{38}$$
$$= w_B(\varphi(y))$$
$$=\varphi^{-1}(w_B)(y).$$

thus $w_B)\left(xyx^{-1}\right)=\varphi^{-1}\ w_B)\ y)$. Therefore *Eqs. (35)-(38)* give us that

$$\varphi^{-1}(r_B)(xyx^{-1}) = r_B(\varphi(xyx^{-1}))$$
$$=r_B(\varphi(x)\varphi(y)\varphi(x^{-1}))$$
$$= r_B(\varphi(x)\varphi(y)\varphi^{-1}(x))$$
$$= r_B(\varphi(y))$$
$$= \varphi^{-1}(r_B)(y),$$

Thus $\varphi^{-1}(B)\in NIFCN(G)$.

Proposition 14. Let $A=\ \mu_A,\ \vartheta_A)\in$ ***IFCN(G)*** and $B=\ \mu_B,\ \vartheta_B)\in$ ***IFCN(G)*** such that $A\sqsubseteq B$.

If $\varphi:G\rightarrow H$ is a group homomorphism, then $\varphi(A)\sqsubseteq\varphi(B)$.

Proof: Let $A=\ \mu_A,\ \vartheta_A)\in IFCN\ G)$ and $B=\ \mu_B,\ \vartheta_B)\in IFCN\ G)$ such that $\mu_A=r_Ae^{iw_A}$ and $\vartheta_A\ x)=r_Ae^{iw_A}$ and $\mu_B=r_Be^{iw_B}$ and $\vartheta_B\ x)=r_Be^{iw_B}$. Using *Proposition 10.* we will have that

$$\varphi\ \ A)=\ \varphi(\mu_A), \varphi(\vartheta_A))=\left(\varphi(r_A)e^{i\varphi(w_A)}), \varphi(r_A)e^{i\varphi(w_A)})\right)\in ICFN\ H),$$

And

$$\varphi\ \ B)=\ \varphi(\mu_B), \varphi(\vartheta_B))=\left(\varphi(r_B)e^{i\varphi(w_B)}), \varphi(r_B)e^{i\varphi(w_B)})\right)\in ICFN\ H).$$

Let $x, y\in H$ and $u, v\in G$ then

$$\begin{aligned}
&\varphi(r_A)(xyx^{-1})=\sup\{r_A(z)\mid z\in G,\varphi(z)=xyx^{-1}\}\\
&=\sup\{r_A(uvu^{-1})\mid u,v\in G,\varphi(u)\ x,\varphi(v)=y\}\\
&\geq\sup\{T(rA(v),r_B(u))\mid\varphi(u)\ x,\varphi(v)=y\}\\
&=T(\sup\{r_A(v)\mid y=\varphi(v)\},\sup\{r_B(u)\mid x=\varphi(u)\})\\
&=T(\varphi(r_A)(y),\varphi(r_B)(x)),
\end{aligned} \tag{39}$$

and so $\varphi\ r_A)\left(xyx^{-1}\right)\geq T\left(\varphi\ r_A)\ y),\varphi\ r_B)\ x)\right)$.

$$\begin{aligned}
&\varphi(w_A)(xyx^{-1})=\sup\{w_A(z)\mid z\in G,\varphi(z)=xyx^{-1}\}\\
&=\sup\{w_A(uvu^{-1})\mid u,v\in G,\varphi(u)\ x,\varphi(v)=y\}\\
&\geq\sup\{\min\{w_A\ v),w_B\ u)\}\mid\varphi(u)\ x,\varphi(v)=y\}\\
&=\min\{\sup\{w_A(v)\mid y=\varphi(v)\},\sup\{w_B(u)\mid x=\varphi(u)\}\}\\
&=\min\{\varphi\ w_A)\ y),\varphi\ w_B)\ x)\},
\end{aligned} \tag{40}$$

and so $\varphi\ w_A)\left(xyx^{-1}\right)\geq\min\{\varphi\ w_A)\ y),\varphi\ w_B)\ x)\}$,

$$\begin{aligned}
&\varphi(r_A)(xyx^{-1})=\sup\{r_A(z)\mid z\in G,\varphi(z)=xyx^{-1}\}\\
&=\inf\{r_A(uvu^{-1})\mid u,v\in G,\varphi(u)\ x,\varphi(v)=y\}\\
&\leq\inf\{S(r_A(v),r_B(u))\mid\varphi(u)\ x,\varphi(v)=y\}\\
&=S(\inf\{r_A(v)\mid y=\varphi(v)\},\inf\{r_B(u)\mid x=\varphi(u)\})\\
&=S(\varphi(r_A)(y),\varphi(r_B)(x)),
\end{aligned} \tag{41}$$

thus $\varphi\ r_A)\left(xyx^{-1}\right)\leq S(\varphi(r_A^{\prime})(y),\varphi(r_B^{\prime})(x))$.

$$\begin{aligned}
&\varphi(w_A)(xyx^{-1})=\inf\{w_A(z)\mid z\in G,\varphi(z)=xyx^{-1}\}\\
&=\inf\{w_A(uvu^{-1})\mid u,v\in G,\varphi(u)\ x,\varphi(v)=y\}\\
&\leq\inf\{\max\{w_A\ v),w_B\ u)\}\mid\varphi(u)\ x,\varphi(v)=y\}\\
&=\max\{\inf\{w_A(v)\mid y=\varphi(v)\},\inf\{w_B(u)\mid x=\varphi(u)\}\}\\
&=\max\{\varphi\ w_A)\ y),\varphi\ w_B)\ x)\},
\end{aligned} \tag{42}$$

and so $\varphi\ w_A^{\prime})\left(xyx^{-1}\right)\leq\max\{\varphi\ w_A^{\prime})\ y),\varphi\ w_B^{\prime})\ x)\}$.

Thus using *Eqs. (39)-(42)* we will have that $\varphi(A)\sqsubseteq\varphi(B)$.

Proposition 15. Let $A=\ \mu_A,\ \vartheta_A)\in IFCN(H)$ and $B=\ \mu_B,\ \vartheta_B)\in IFCN(H)$ such that $A\sqsubseteq B$.

If $\varphi:G\to H$ is a group homomorphism, then $\varphi^{-1}(A)\sqsubseteq\varphi^{-1}(B)$.

Proof: Let $A=\ \mu_A,\ \vartheta_A)\in IFCN\ H)$ and $B=\ \mu_B,\ \vartheta_B)\in IFCN\ H)$ such that $\mu_A=r_Ae^{iw_A}$ and $\vartheta_A\ x)=r_Ae^{iw_A}$ and $\mu_B=r_Be^{iw_B}$ and $\vartheta_B(x)=r_B^{\prime}e^{iw_B^{\prime}}$. Using *Proposition 11.* we will have that

$$\varphi^{-1}\ A)=\left(\varphi^{-1}(\mu_A),\varphi^{-1}(\vartheta_A)\right)=\left(\varphi^{-1}(r_A)e^{i\varphi^{-1}(w_A)},\varphi^{-1}(r_A)e^{i\varphi^{-1}(w_A)}\right)\in ICFN\ G).$$

And

$$\varphi^{-1}\ B)=\left(\varphi^{-1}(\mu_B),\varphi^{-1}(\vartheta_B)\right)=\left(\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)},\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)}\right)\in ICFN\ G).$$

Let $x,y\in G$, then

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$$\begin{aligned}&\varphi^{-1}(r_A)(xyx^{-1}) = r_A(\varphi(xyx^{-1}))\\&=r_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\&= r_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\&\geq T(r_A(\varphi(y)), r_B(\varphi(x)))\\&=T(\varphi^{-1}(r_A)(y), \varphi^{-1}(r_B)(x)),\end{aligned} \tag{43}$$

Then $\varphi^{-1}\ r_A)\left(xyx^{-1}\right)\geq T\left(\varphi^{-1}\ r_A)\ y), \varphi^{-1}\ r_B)\ x)\right)$.

$$\begin{aligned}&\varphi^{-1}(w_A)(xyx^{-1}) = w_A(\varphi(xyx^{-1}))\\&=w_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\&=w_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\&\geq \min\{w_A\left(\varphi\ y)\right), w_B\left(\varphi\ x)\right)\}\\&=\min\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\},\end{aligned} \tag{44}$$

thus $\varphi^{-1}\ w_A)\left(xyx^{-1}\right)\geq \min\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\}$.

$$\begin{aligned}&\varphi^{-1}(r_A)(xyx^{-1}) =r_A(\varphi(xyx^{-1}))\\&=r_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\&=r_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\&\leq S(r_A(\varphi(y)), r_B(\varphi(x)))\\&=S(\varphi^{-1}(r_A)(y), \varphi^{-1}(r_B)(x)),\end{aligned} \tag{45}$$

so $\varphi^{-1}\ r_A)\left(xyx^{-1}\right)\leq S\left(\varphi^{-1}\ r_A)\ y), \varphi^{-1}\ r_B)\ x)\right)$.

$$\begin{aligned}&\varphi^{-1}(w_A)(xyx^{-1})=w_A(\varphi(xyx^{-1}))\\&=w_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\&=w_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\&\leq \max\{w_A\left(\varphi\ y)\right), w_B\left(\varphi\ x)\right)\}\\&= \max\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\},\end{aligned} \tag{46}$$

thus $\varphi^{-1}\ w_A)\left(xyx^{-1}\right)\leq \max\left\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\right\}$.

Thus *Eqs. (43)-(46)* give us that $\varphi^{-1}(A)\sqsubseteq\varphi^{-1}(B)$.

7 | Conclusion and Open Problem

In this study, intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm sare defined and investigated some properties of them. Later, the inverse, composition, intersection and normality of them are introduced and we proved some basic new results and present some properties of them. Now one can investigate intuitionistic fuzzy complex submodules with respect to t-norm T and s-norm S as we did and this can be an open problem. We would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

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Conflicts of Interest

The author has seen and agreed with the contents of the manuscript and there is no financial interest to report and certifies that the submission is original work and is not under review at any other publication.

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